Mathematics Learning Instrument: Development of an Algebra Concept Inventory to Measure Metric Sense Conceptual Understanding

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Abstract

Mathematics learning outcomes have specific cognitive development implications [2], [9]. Empirical studies in mathematics learning have contributed greatly in our understanding of mathematical spaces, structures, and learning. Key contributions in 'structure sense' studies build upon a theoretical manuscript detailing a conceptualization of 'thinking mathematically' using symbols (i.e. symbol sense) [1], [6], [12], [13]. Likewise, this theoretical manuscript reports on (a) a definition of 'metric sense' [9], (b) specific roles of mini-experts in Second Generation Instructional Design (ID2) [11], and, (c) creates a mathematics instrument using 'metric sense' to inventory subject specific domain knowledge base items [5]. This paper details horizontal integration of five specific mathematics knowledge base elements predetermined by a metric space [5]. Recommendations are made for further mathematics learning investigations using the instrument.

1. Introduction

This theoretical manuscript presents a definition of *metric sense* using a metric space [12]. Mathematics learning requires a continual examination of conceptual understanding while executing each step of procedural approaches to problem solving [2]. Recent research has examined the effect of using brackets in procedural processes to group structures according to an order of operations [6]. Other studies compare written forms of expressions to examine student comprehension of structure (i.e. *structure sense*) [1], [10].

In this paper it is contended that procedural processes are concurrent with structure comprehension in making sense of mathematics [10], [13]. Theoretically, the inherent mathematical structure of a *metric space* will define the algebraic structures mathematics students engage when using procedures to rearrange groupings of symbols to solve problems [12]. Five horizontal integrated algebra knowledge base elements are hypothesized as concurrently necessary to simplify expressions, solve equations, and build skills in mathematics learning [2], [5]. The distance function relating elements of a metric space (e.g. variable representations among symbols in algebra) is used in an analysis to justify each integrated algebra knowledge base element. Findings include an Algebra Concept Inventory to Measure Metric Sense and two examples which extend the research constructs to post algebra structures (see Tables 3, 4, 2, and Figures 1a, 1b respectively). The instrument is consistent with current lines of inquiry in mathematics learning and may compliment instrumentation developed in studies in *Newtonian physics* [3], [4].

2. Metric Space and Metric Sense

The most familiar *metric space* used to develop elementary algebraic skills is the Euclidean Space [8]. The Euclidean plane (i.e. Cartesian Plane **R**x**R**) is used as a representation for learning algebraic concepts where two *real* number lines intersect at an origin. The representation provides access to instruction on two variables x and y. The following definition of a metric space theoretically supports the development of instrument design and element horizontal integration into a specific mathematics *domain knowledge base* [5], [12].

A *metric space* is a set **R** with a distance function (the metric d) that, for every two points x,y in **R**, gives the distance between them as a nonnegative real number d(x,y). A metric space must also satisfy

1. $d(x,y) = 0$ if and only if $x = y$
2. $d(x,y) = d(y,x)$ and
3. the triangle inequality $d(x,y) + d(y,z) >= d(x,z)$.

"A *Euclidian Space* may be viewed as a *vector space* with the *usual metric* (distance) explored in secondary mathematics education. This focus on mathematical conceptual understanding, a refinement of *structure sense* within *mathematical thinking*, herein termed *metric sense*; is hypothesized to be instrumental in

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building empirical knowledge in" … mathematics instruction … "and the assessment of learning outcomes" [8], [9].

3. Metric sense as horizontal integrated algebra knowledge base elements

A purpose of this paper is to specify algebra knowledge base elements impacting future knowledge production in mathematics learning. Measuring *metric sense* cognitive skills is inherently measuring proficiency in identifying and correctly interpreting *metric space* elements [9], [12]. The following delineates five measurable constructs of metric sense conceptualization (MS1-MS5) throughout each step in an equation and its procedural solution (see Table 1).

1. MS1 - identify all the unary, binary, or other operations
2. MS2 – identify the *main operation* (MO) of an expression within an equation; utilizing all the expression, not missing anything
3. MS3 - correctly execute the inverse operation resulting in an equivalent equation
4. MS4 – identify all positive and negative symbols in an expression as either only part of a coefficient of a term, or concurrently as a coefficient symbol and a binary operation (i.e. sum and difference)
5. MS5 – identify in writing, the hierarchy order of operations beginning with the main operation (MO), throughout the entire expression, detailing all operands of unary and binary operations.

Table 1. Algebraic example identifying all metric sense elements in each solution step

$\frac{+x^2-4}{+10}=+6$ MS3 {apply inverse, multiply}	MS2 MO quotient MS5 Quotient of (difference and 10) ↓ (Square and 4) ↓ x	MS1 - Two binary operations (-, /) and one unary operation (square) MS4 – Three positive symbols as coefficients on terms; one negative symbol as symbol on 4 having dual purpose as binary operation difference/subtraction (total 4)
$+x^2-4=+60$ MS3 {apply inverse add}	MS2 MO difference MS5 Difference (Square and 4) ↓ x	MS1 - One binary operation (-) and one unary operation (square) MS4 – Two positive symbols as coefficients on terms; one negative symbol as symbol on 4 having dual purpose as binary operation difference/subtraction (total 3)
$+x^2=+64$ MS3 {apply inverse $\sqrt{x^2}=\sqrt{64}$}	MS2 MO square MS5 (Square) ↓ x	MS1 - one unary operation MS4 – Two positive symbols as coefficients on terms; no positive or negative symbols having dual purpose as binary operations (total 2)
$+x=+8$ *		MS4 – Two positive symbols as coefficients on terms; no positive or negative symbols having dual purpose as binary operations (total 2)

Note. * Positive result for illustration

4. A relationship between metric sense knowledge elements and a mathematical metric space

The hypothesis of this manuscript is: a student who has a high level of metric sense in the five horizontal integrated algebra knowledge base elements has a knowledge base necessary to simplify expressions, solve equations, and build knowledge in a metric space setting.

5. Theoretical analysis of the hypothesis

Represented here are theoretical justifications for the hypothesis [12].

MS1, MS4: A metric space requires, for every two points x,y in **R**, the triangle inequality d(x,y) + d(y,z) >= d(x,z) holds. Therefore, the summation, and its inverse binary operation, difference, must be identifiable as independent or concurrent uses of the + and – symbols respective of an element's distance from either side of zero on a *real* number line.

MS2, MS5: Novotná and Hoch [12, p. 95] find "An important feature of structure sense is the substitution principle, which states that if a variable or parameter is replaced by a compound term (product or sum), or if a compound term is replaced by a parameter, the structure remains the same." Therefore, a *main operation* (MO) of an expression within an equation, utilizing all the expression, not missing anything; incorporates possible compound terms where in a metric space, for every two points x, y in **R**, gives the distance between them, d(x, y).

MS3: The relationship between an operation's identity element and inverse elements of a metric space are identifiable for every element in the continuous set **R** [8].

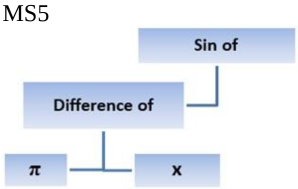
An extended application of the five measurable components of metric sense conceptualization, with the same theoretical justifications, can be applied to post algebra mathematics learning [12]. An application of conceptualizing *metric sense* in a trigonometric equation is provided (see Table 2). Although introduced in algebra, it cannot be over emphasized that continual conceptualization and persistent application of the five horizontal integrated algebra knowledge base elements is critical to mathematics learning throughout secondary and tertiary education.

Table 2. Trigonometric example identifying metric sense knowledge base elements

<table>
<tr>
<td>$+\sin(+\pi - x) = -2$

Normally:
$\sin(\pi - x) = -2$

MS3 {apply inverse $\sin^{-1}$}</td>
<td>MS2 MO sin function {unary}

MS5

</td>
<td>MS1 - One binary operation (-) and one unary operation, sin ( )

MS4 – Three positive/negative symbols as coefficients on terms, One negative symbol as a symbol on x with dual purpose as a binary operation difference (total 4)</td>
</tr>
</table>

Note. $\sin^{-1}$ is the inverse function of the sine function; first solution step provided for illustration

An additional application of metric sense and structure sense is presented for student learning on the proportional minimum sample size n formula from a familiar margin of error (E) equation (see Figure 1a and 1b). The classroom presentation episode is a successful example of turning theory into practice.

6. A mathematics instrument using 'metric sense' to inventory algebra subject specific domain knowledge base elements

Expressions typical in linear and quadratic equations are analyzed in ten questions of an Algebra Concept Inventory to Measure Metric Sense (see Table 3 and 4). Level appropriate vocabulary is integrated in the instrument which is conventionally familiar to beginning algebra students. Some latent coefficient symbols are present to accentuate *structure sense* knowledge base element identification while using the instrument (i.e. MS4, see Tables 1- 4). It is recommended that instructors use the instrument customized and adapted to their individual needs. Instructors wishing to use the instrument to find a significance between data with other assessments in algebra may need to determine pre and post testing variables according to their particular school setting. Additional delineations may need to be specified when reporting on findings using the instrument.

Table 3. Algebra Concept Inventory to Measure Metric Sense p. 1

Answer questions 1-5 using the equation below:

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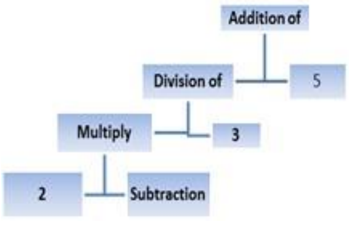
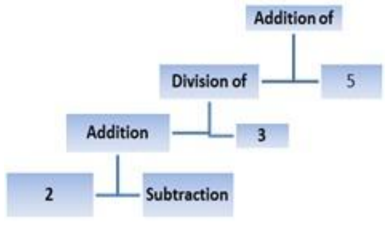
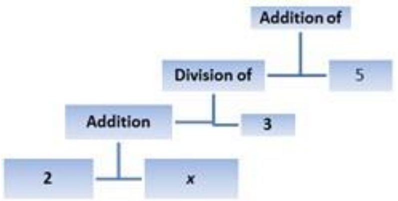
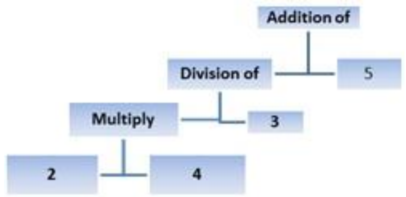
$$\frac{+2(+x-4)}{+3}+5=0$$ Normally $\frac{2(x-4)}{3}+5=0$

1. Which is the main operation of the expression on the **left side** of the equation; utilizing **all** of the left expression, not missing anything, considering the order of operations
   A. + symbol before the x
   B. ÷ division bar
   C. + symbol before the 5
   D. - symbol before the 4
2. Which is correct about the expression on the **left side** of the equation
   A. **four** positive symbols on terms; **one** negative symbol also as subtraction (total **five**)
   B. **one** positive symbol on terms; **one** negative symbol also as subtraction; **three** positive symbols also as addition (total **five**)
   C. **two** positive symbols on terms; **one** negative symbol also as subtraction; **two** positive symbols also as addition (total **five**)
   D. **three** positive symbols on terms; **one** negative symbol also as subtraction; **one** positive symbol also as addition (total **five**)
3. Which is accurate about the expression on the **left side** of the equation, considering the order of operations

   A. 

   B. 

   C. 

   D. 

4. Which is accurate about the expression on the **left side** of the equation
   A. It has one binary operation
   B. It has four binary operations
   C. It has five binary operations
   D. It has no binary operations
5. What are the steps in solving the equation by reversing the main operation of the expression on the left side of the equation;
   A. First subtract each side by 5 followed by multiplying by 3 as the next step
   B. First subtract each side by 5 followed by adding by 4 as the next step
   C. First subtract each side by 4 followed by multiplying by 3 as the next step
   D. First add each side by 5 followed by dividing by 4 as the next step

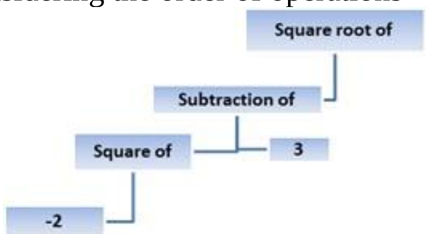
Note. Coefficient symbols are shown and introductory vocabulary used for inventory instrument measurement.

Table 4. Algebra Concept Inventory to Measure Metric Sense p. 2

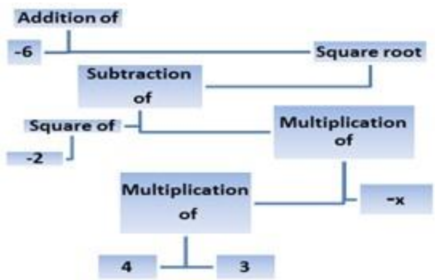
Answer questions 6-10 using the equation below:

$$-6+\sqrt{(-2)^2-4(+3)(-x)}=0$$ Normally $-6+\sqrt{(-2)^2-4(3)(-x)}=0$

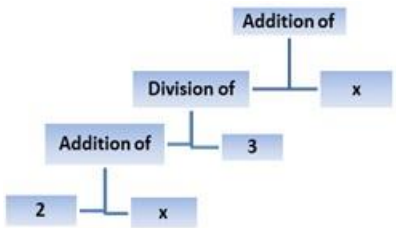
6. Which is the main operation of the expression on the **left side** of the equation; utilizing **all** of the left expression, not missing anything, considering the order of operations
   - A. - symbol before the 4
   - B. + symbol after the 6
   - C. ^{2} symbol (square)
   - D. $\sqrt{\ }$ symbol (square root)
7. Which is correct about the expression on the **left side** of the equation
   - A. **one** negative symbol on terms; **one** positive symbol on terms; **two** positive symbols also as addition; **two** negative symbols also as subtraction (total **six**)
   - B. **two** negative symbols on terms; **one** positive symbol on terms; **one** positive symbol also as addition; **two** negative symbols also as subtraction (total **six**)
   - C. **three** negative symbols on terms; **one** positive symbol on terms; **one** positive symbol also as addition; **one** negative symbol also as subtraction (total **six**)
   - D. **one** negative symbol on terms; **one** positive symbol on terms; **one** positive symbol also as addition; **three** negative symbols also as subtraction (total **six**)
8. Which is accurate about the expression on the **left side** of the equation, considering the order of operations



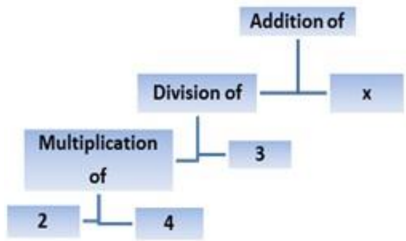
A.



B.



C.



D.

9. What is accurate about the expression on the **left side** of the equation
   - A. It has 6 binary and 1 unary operation
   - B. It has 4 binary and 1 unary operations
   - C. It has 1 binary and 4 unary operations
   - D. It has 4 binary and 2 unary operations

10. What are the steps in solving the equation by reversing the main operation of the expression on the left side of the equation;
   - A. First add each side by 6 followed by squaring each side as the next step
   - B. First square each side followed by adding each side by 6
   - C. First add each side by -2 followed by squaring each side as the next step
   - D. First square each side followed by adding each side by 2

Note. Coefficient symbols are shown and introductory vocabulary used for inventory instrument measurement.

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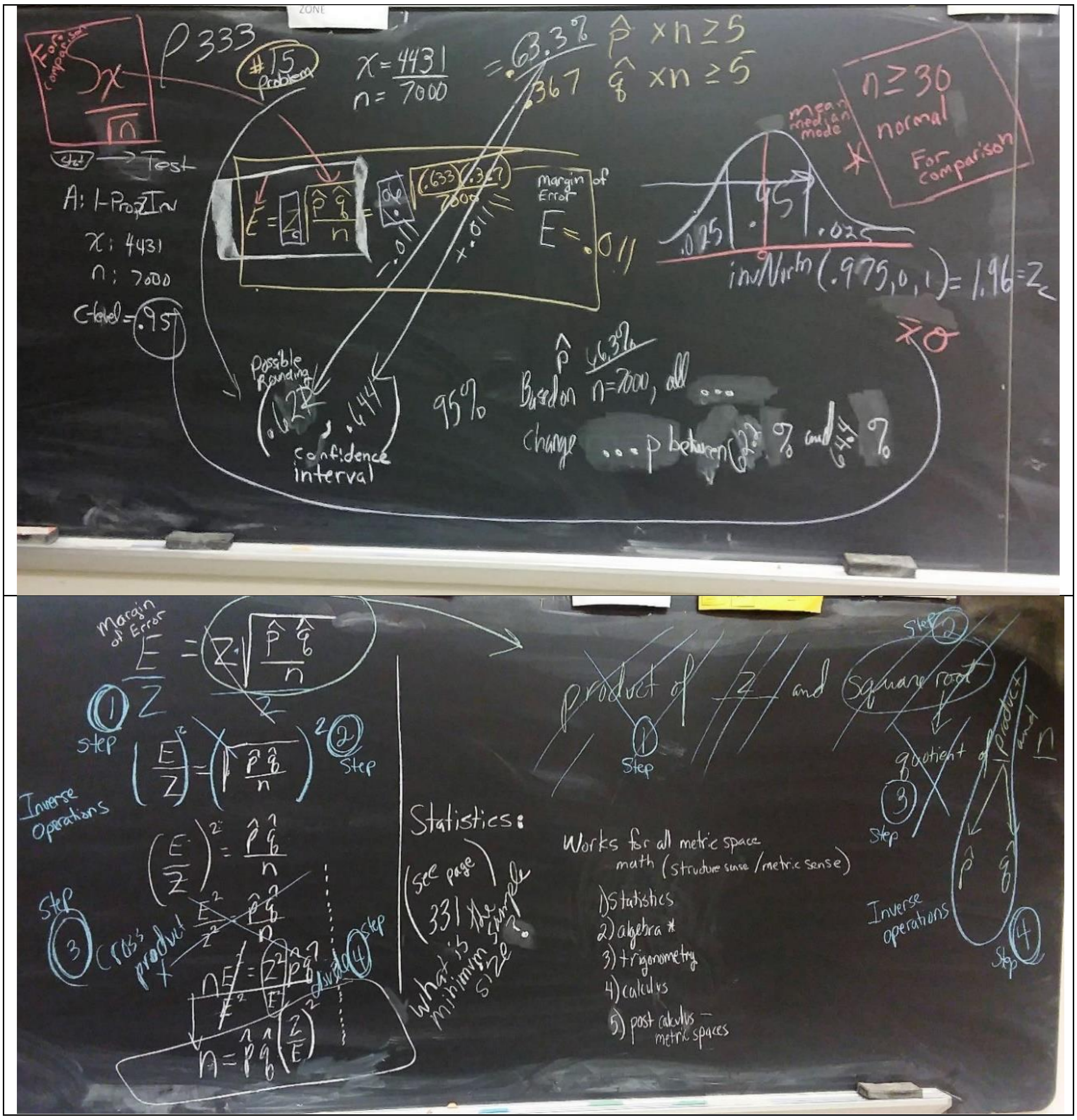

Figure 1a and 1b. Metric sense instruction as constructs on confidence intervals for proportions

7. Conclusions

There are specific cognitive skills required in representing and explaining *structure sense* within the mathematical construct of a *Euclidian Space* [8], [9]. Students' skills can be quantified for potential manipulations in algebraic equations and throughout each step in reaching solutions. These theoretical findings can be extended throughout mathematical expressions common in secondary and tertiary education. A quantitative instrument for researching mathematics learning through contextually rich procedural analysis (e.g. detailing step by step solutions) can be created (see table 3 and 4). This instrument will fill a void in tools needed for *mathematical thinking* analysis [13]. Inclusion of *structure sense* analysis in empirical studies is consistent with goals of *second generation instructional design* constructs, approaching a moderation of constructivism [5], [7], [11]. As a result of the hypothesis of this theoretical manuscript, it is found that measuring the mathematics learning construct of *metric sense* is beneficial in researching mathematics instruction and assessment of learning outcomes [2], [9], [13].

8. Implications for further research

This manuscript provides the Algebra Concept Inventory to Measure Metric Sense (i.e. quantitative instrument) to measure mathematics conceptual understanding using five measurable constructs of metric sense. This inventory instrument was inspired by reviewing research manuscripts using the Force Concept Inventory (FCI) instrument [9]. The FCI test use in empirical studies has initiated substantial contributions to the line of inquiry in physics education [3], [4]. Instrumentation measuring the five constructs of *metric sense* conceptualization (MS1-MS5) has the potential to contribute similarly to the line of inquiry in mathematics education.

It is recommended that future research include correlation studies between instrument data and results from algebra standards testing instrumentation commonly found within secondary education. A correlation study is in progress by the author and a report of findings will be presented in a future publication. Included in mathematics learning mixed method studies, the instrument may provide evidence of relationships across data sources or research design approaches. It is also recommended the instrument be used with the FCI instrument to measure scientific reasoning ability; adding some clarity to unexplained negative correlation findings between specific physics learner sample subgroups [3].

9. References

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